

ON THE DIRECT SUMMAND CONJECTURE AND ITS DERIVED VARIANT

BHARGAV BHATT

ABSTRACT. André recently gave a beautiful proof of Hochster’s direct summand conjecture in commutative algebra using perfectoid spaces; his two main results are a generalization of the almost purity theorem (the perfectoid Abhyankar lemma) and a construction of certain faithfully flat extensions of perfectoid algebras where “discriminants” acquire all p -power roots.

In this paper, we explain a quicker proof of Hochster’s conjecture that circumvents the perfectoid Abhyankar lemma; instead, we prove and use a quantitative form of Scholze’s Hebbbarkeitssatz (the Riemann extension theorem) for perfectoid spaces. The same idea also leads to a proof of a derived variant of the direct summand conjecture put forth by de Jong.

1. INTRODUCTION

The first goal of this paper is to give an alternative and shorter proof the following recent result of André, settling the direct summand conjecture:

Theorem 1.1 (André). *Let $i : A_0 \hookrightarrow B_0$ is a finite extension of noetherian rings. Assume that A_0 is regular. Then the inclusion i is split as an A_0 -module map.*

When A_0 has characteristic 0, Theorem 1.1 is easy to prove using the trace map. When $\dim(A_0) \leq 2$, one can prove Theorem 1.1 using the Auslander-Buchsbaum formula. Hochster conjectured the general case in 1969, and proved it when A_0 has characteristic p in [Ho1]. The first general result in mixed characteristic was Heitmann’s [He], settling the case of dimension 3. More on the history of this conjecture and its centrality amongst the ‘homological conjectures’ in commutative algebra can be found in [Ho3]. The result above is proven by André [An2] using [An1].

In this paper, we give a proof of Theorem 1.1 that avoids [An1] (and is independent of [An2] in terms of exposition). Our approach also adapts to yield the following derived variant, which was conjectured by Johan de Jong in the course of the author’s thesis work [Bh1, Bh3] as a path towards understanding the direct summand conjecture:

Theorem 1.2. *Let A_0 be a regular noetherian ring. Let $f_0 : X_0 \rightarrow \operatorname{Spec}(A_0)$ be a proper surjective map. Then the map $A_0 \rightarrow R\Gamma(X_0, \mathcal{O}_{X_0})$ splits in the derived category $D(A_0)$.*

When A_0 has characteristic 0, this result is due to Kovács [Ko], and is deduced from the fact that $\operatorname{Spec}(A_0)$ has rational singularities; see also [Bh1, Theorem 2.12]. The characteristic p case follows from [Bh1, Theorem 1.4 & Example 2.3]. To the best of our knowledge, in mixed characteristic, Theorem 1.2 is new even when $\dim(A_0) = 2$.

Assumption 1.3. In the rest of the introduction, primarily for notational ease, we assume that $A_0 := \widehat{W[x_1, \dots, x_d]}$ is the p -adic completion of a polynomial ring over an unramified dvr W of mixed characteristic $(0, p)$; there is a standard reduction of Theorem 1.1 to mild variants of such an A_0 , so not much generality is lost.

1.1. The strategy of André’s proof. André’s proof of Theorem 1.1 uses perfectoid spaces [Sc1]. To see why this is natural, we first informally outline the main idea, adapted from [Bh2], in the special case where $A_0[\frac{1}{p}] \rightarrow B_0[\frac{1}{p}]$ is étale. The crucial input is Faltings’ almost purity theorem [Fa4]¹, which asserts: if $A_{\infty,0}$ is the p -adic completion of $A_0[x_i^{\frac{1}{p^\infty}}, p^{\frac{1}{p^\infty}}]$, then the p -adic completion of the integral closure B_∞ of B_0 in $B_0 \otimes_{A_0} A_{\infty,0}[\frac{1}{p}]$ is *almost* finite

¹Faltings’ theory of almost mathematics, in one incarnation, describes commutative algebra over a ring V equipped with a nonzerodivisor $f \in V$ that admits arbitrary p -power roots. More precisely, one works in the quotient of the abelian category of V -modules by the Serre category of all $(f^{\frac{1}{p^\infty}})$ -torsion modules. The study of such ‘almost modules’ was inspired by Tate’s [Ta], and led Faltings to prove fundamental results in p -adic Hodge theory [Fa1, Fa2, Fa3, Fa4]. Following Faltings’ work, a systematic investigation of almost mathematics was carried by Gabber and Ramero [GR]. The relevance of almost mathematics to the direct summand conjecture seems to be first suggested by Roberts [GR, §0].

étale over $A_{\infty,0}$ with respect to the ideal $(p^{\frac{1}{p^\infty}}) \subset A_{\infty,0}$. Concretely, the algebraic obstructions to $A_{\infty,0} \rightarrow B_\infty$ being finite étale — such as the cokernel of the trace map $B_\infty \rightarrow A_{\infty,0}$ or the Ext^1 -class measuring the failure of $A_{\infty,0} \rightarrow B_\infty$ to split — are killed by $p^{\frac{1}{p^k}}$ for all $k \geq 0$, and are thus quite ‘small’. The faithful flatness of $A_0 \rightarrow A_{\infty,0}$ and the noetherianness of A_0 then let one conclude that $A_0 \rightarrow B_0$ must actually split. Summarizing, the key ideas are:

- (1) The construction of the faithfully flat extension $A_0 \rightarrow A_{\infty,0}$.
- (2) The almost splitting after base change to $A_{\infty,0}$ coming from the almost purity theorem.

It is now easy to see why perfectoid spaces provide a natural conceptual home for this proof: the ring $A_{\infty,0}$ in (1) is (integral) perfectoid², and the almost purity theorem invoked in (2) is a general fact valid for finite extensions of any perfectoid algebra that are étale after inverting p (due to Kedlaya-Liu [KL] and Scholze [Sc1]).

André’s proof of Theorem 1.1 follows a similar outline to the one sketched above. The first major difference is that $A_{\infty,0}$ is replaced by a larger perfectoid extension A_∞ of $A_{\infty,0}$ coming from the following remarkable construction:

Theorem 1.4 (André). *Fix $g \in A_0$. Then there exists a map $A_{\infty,0} \rightarrow A_\infty$ of integral perfectoid algebras that is almost faithfully flat modulo p such that the element $g \in A_0$ admits a compatible system of p -power roots $g^{\frac{1}{p^k}}$ in A_∞ .*

André’s proof of Theorem 1.4 relies crucially on perfectoid geometry, and is explained in §2. For the application to Theorem 1.1, one chooses $g \in A_0$ to be a discriminant, i.e., an element g such that $A_0[\frac{1}{g}] \rightarrow B_0[\frac{1}{g}]$ is finite étale. The flatness assertions in Theorem 1.4 then reduce us almost splitting the base change $A_\infty \rightarrow B_0 \otimes_{A_0} A_\infty$.

To construct an almost splitting over A_∞ , André proves a much stronger result, which forms the subject of [An1]: he generalizes the almost purity theorem to describe extensions of A_∞ that are étale after inverting g (almost purity corresponds to $g = p$). The output is *roughly* that the integral closure B_∞ of $B_0 \otimes_{A_0} A_\infty$ in $B_0 \otimes_{A_0} A_\infty[\frac{1}{g}]$ is almost finite étale over A_∞ , where ‘almost mathematics’ is measured with respect to $((pg)^{\frac{1}{p^\infty}}) \subset A_\infty$; the precise statement is more subtle, and we do not formulate it here as we do not need it.

1.2. The strategy of our proof. Our proof uses Theorem 1.4. Thus, the task is to (almost) split $A_0 \rightarrow B_0$ after base change to the ring A_∞ arising from Theorem 1.4. For this, we again use perfectoid geometry. More precisely, for each $n \geq 1$, the general theory gives us the perfectoid ring $A_\infty \langle \frac{p^n}{g} \rangle$ of bounded functions on the rational subset

$$U_n := \{x \in X \mid |p^n| \leq |g(x)|\}$$

of the perfectoid space X associated to A_∞ . These rings naturally form a projective system as n varies (since $U_n \subset U_{n+1}$), and can be almost described very explicitly: to get $A_\infty \langle \frac{p^n}{g} \rangle$, one formally adjoins $\frac{p^n}{g}$ and its p -power roots to A . Their main utility to us³ is that g divides p^n in $A_\infty \langle \frac{p^n}{g} \rangle$, so $A_0 \rightarrow B_0$ becomes finite étale after base change to $A_\infty \langle \frac{p^n}{g} \rangle[\frac{1}{p}]$ for any $n \geq 0$; the almost purity theorem then kicks in to show that for each $n \geq 0$, the base change of $A_0 \rightarrow B_0$ to the perfectoid algebra $A_\infty \langle \frac{p^n}{g} \rangle$ is almost split. To descend the splitting to A_∞ , we prove the following quantitative form of Scholze’s Riemann extension theorem [Sc2, Proposition II.3.2].

Theorem 1.5. *Fix an integer $m \geq 0$. The natural map of pro-systems*

$$\{A_\infty/p^m\}_{n \geq 1} \rightarrow \{A_\infty \langle \frac{p^n}{g} \rangle / p^m\}_{n \geq 1} \quad (1)$$

is an almost-pro-isomorphism with respect to $(pg)^{\frac{1}{p^\infty}}$, i.e., for each $k \geq 0$, the pro-system of kernels and cokernels is pro-isomorphic to a pro-system of $(pg)^{\frac{1}{p^k}}$ -torsion modules.

Remark 1.6. On taking limits over n and m in Theorem 1.5, one obtains the following statement from [Sc2, Proposition II.3.2]: any bounded function on the Zariski open set $\{x \in X \mid g(x) \neq 0\} = \cup_n U_n \subset X$ almost extends to X . This explains why one views this result as a perfectoid analog of the classical Riemann extension theorem in complex geometry. A similar result in rigid geometry was proven by Bartenwerfer [Ba].

²A precise definition is given in §1.4. The crucial consequence of a ring R being perfectoid is that the Frobenius $R/p \rightarrow R/p$ is surjective, and has a large but controlled kernel. Under suitable completeness and torsionfreeness hypotheses, this leads to the existence of lots of elements that admit arbitrary p -power roots, such as the elements p and x_i in $R = A_{\infty,0}$.

³The idea of using this tower of rings to study A_∞ also comes from [An1].

Theorem 1.5 says that the limiting isomorphism $A_\infty \xrightarrow{a} \lim A_\infty \langle \frac{p^n}{g} \rangle$ holds true for ‘diagrammatic’ reasons, and thus is also true after applying A_∞ -linear functors, such as $\mathrm{Ext}_{A_\infty}^i(N, -)$ for an A_∞ -module N , to both sides of (1) and then taking limits. Using this ingredient, the proof of Theorem 1.1 proceeds along the lines sketched above. Thus, the proof can be summarized as follows: pass from A_0 to A_∞ using Theorem 1.4 to ensure this passage is lossless, pass from A_∞ to $A_\infty \langle \frac{p^n}{g} \rangle$ to push all the ramification into characteristic p , construct an almost splitting over $A_\infty \langle \frac{p^n}{g} \rangle$ using almost purity, and finally take a limit over n to get an almost splitting over A_∞ thanks to Theorem 1.5.

To prove Theorem 1.2, we proceed analogously. First, assume that f_0 ramifies only in characteristic p , i.e., $f_0[\frac{1}{p}]$ is finite étale. Again, it suffices to construct the splitting after going up to a faithfully flat integral perfectoid extension of A_0 (such as the ring $A_{\infty,0}$ above). After such a base change, the almost purity theorem and a general vanishing theorem of Scholze settle the question. In general, one first reduces to f_0 being generically finite, and finite étale after inverting some $g \in A_0$. This case is then deduced from preceding special case using Theorem 1.4 and Theorem 1.5, exactly as was explained above for Theorem 1.1.

1.3. Layout. We begin in §2 by recalling André’s proof of Theorem 1.4. Theorem 1.5 is proven in §4; this depends on the notion of almost mathematics of pro-systems, which is briefly developed in §3. With these ingredients in place, Theorems 1.1 and 1.2 are proven in §5 and §6 respectively.

1.4. Notation. We freely use the language of perfectoid spaces and almost mathematics. Occasionally, we use almost mathematics with respect to different ideals in the same ring; thus we always specify the relevant ideal, sometimes at the beginning of each section. The letter K denotes a perfectoid field⁴, and $K^\circ \subset K$ is the ring of integers. We fix an element $t \in K^\circ$ which admits arbitrary p -power roots $t^{\frac{1}{p^k}}$, and such that $|t| = |p|$ if K has characteristic 0. A K° -algebra A is called *integral perfectoid* if it is flat, t -adically complete, satisfies⁵ $A = A_*$, and satisfies the following: Frobenius induces an isomorphism $A/t^{\frac{1}{p}} \simeq A/t$. The category of such algebras is equivalent to usual category of perfectoid K -algebras by [Sc1, Theorem 5.2]; the functors are $A \mapsto A[\frac{1}{t}]$ and $R \mapsto R^\circ$ respectively. Given an integral perfectoid K° -algebra A , the associated perfectoid space $\mathrm{Spa}(A[\frac{1}{t}], A)$ is abusively denoted $\mathrm{Spa}(A[\frac{1}{t}])$ instead.

Acknowledgements. This paper is inspired by André’s preprints [An1, An2]; I thank heartily him for sharing them, and for his kind words about this paper. I am also very grateful to Peter Scholze for patiently explaining basic facts about perfectoid spaces, for discussions surrounding Theorem 1.4, and for convincing me that ‘almost-pro-isomorphism’ is a better name than ‘pro-almost-isomorphism’ in §3. I am equally indebted to my former PhD advisor Johan de Jong for talking through the contents of this manuscript, and his very prescient suggestion (slightly over seven years ago!) that I pursue the mathematics surrounding Theorem 1.2. I was partially supported by NSF Grant DMS #1501461 and a Packard fellowship during the preparation of this work.

2. ADJOINING ROOTS OF THE DISCRIMINANT

Notation 2.1. Let A be an integral perfectoid K° algebra. Fix $g \in A$. Set $X := \mathrm{Spa}(A[\frac{1}{t}])$ and $Y := \mathrm{Spa}(A \langle T^{\frac{1}{p^\infty}} \rangle [\frac{1}{t}])$; these are perfectoid spaces. All occurrences of almost mathematics in this section are with respect to $t^{\frac{1}{p^\infty}}$.

The main goal of this section is to construct an almost faithfully flat extension $A \rightarrow A_\infty$ of perfectoid algebras such that g acquires arbitrary p -power roots in A_∞ . For this, we essentially set $T = g$ in $A \langle T^{\frac{1}{p^\infty}} \rangle$. More precisely, to get a perfectoid algebra, we approximate bounded functions on the Zariski closed space

$$Z := V(T - g) := \{y \in Y \mid T(y) = g(y)\} \subset Y$$

using bounded functions on rational open neighbourhoods

$$Y \langle \frac{T - g}{t^\ell} \rangle := \{y \in Y \mid |T(y) - g(y)| \leq |t^\ell|\} \subset Y$$

of Z for varying integers ℓ .

⁴At first pass, not much is lost if one simply sets $K := \widehat{\mathbf{Q}_p(p^{\frac{1}{p^\infty}})}$ in characteristic 0 (with $t = p$), and $K := \widehat{\mathbf{F}_p((t^{\frac{1}{p^\infty}}))}$ in characteristic p .

⁵Concretely, the assumption $A = A_*$ means that if $f \in A[\frac{1}{t}]$ is such that $t^{\frac{1}{p^k}} \cdot f \in A$ for all $k \geq 0$, then $f \in A$.

Definition 2.2. Set A_∞ to be the integral perfectoid⁶ ring of functions on the Zariski closed subset of Y defined by the ideal $(T - g)$, in the sense of [Sc2, §II.2]. Explicitly, we have

$$A_\infty = \widehat{\operatorname{colim}_{\ell \in \mathbb{N}} B_\ell},$$

where the completion is t -adic, and

$$B_\ell := \mathcal{O}_Y^+(Y \langle \frac{T-g}{t^\ell} \rangle) := A \langle T^{\frac{1}{p^\infty}} \rangle \langle \frac{T-g}{t^\ell} \rangle$$

is the displayed integral ring of functions on the rational subset $Y \langle \frac{T-g}{t^\ell} \rangle \subset Y$.

Note that $T = g$ in A_∞ as $(T - g)$ is divisible by t^ℓ in B_ℓ , and thus in A_∞ , for all ℓ . Thus, g has a distinguished system of p -power roots $g^{\frac{1}{p^k}} := T^{\frac{1}{p^k}}$ in A_∞ . The main theorem is (see [An2, §2.5]):

Theorem 2.3 (André). *The map $A \rightarrow A_\infty$ is almost faithfully flat modulo t .*

Proof. It suffices to show that for any fixed $\ell \geq 0$, the map $A \rightarrow B_\ell$ is almost faithfully flat modulo t^ϵ for some $\epsilon = \epsilon(\ell) > 0$. For fixed ℓ , Scholze's approximation lemma [Sc1, Corollary 6.7] gives an $f \in (A \langle T^{\frac{1}{p^\infty}} \rangle)^\flat$ such that

- (1) $f^\sharp \equiv T - g \pmod{t^{\frac{1}{p}}}$.
- (2) We have an equality $Y \langle \frac{T-g}{t^\ell} \rangle = Y \langle \frac{f^\sharp}{t^\ell} \rangle$ of subsets of Y .

The explicit description of $\mathcal{O}_Y^+(Y \langle \frac{f^\sharp}{t^\ell} \rangle)$ from [Sc1, Lemma 6.4] identifies B_ℓ (almost) with the t -adic completion of

$$\operatorname{colim}_k \left(A \langle T^{\frac{1}{p^\infty}} \rangle [u^{\frac{1}{p^k}}] / ((u \cdot t^\ell)^{\frac{1}{p^k}} - (f^\sharp)^{\frac{1}{p^k}}) \right) = A \langle T^{\frac{1}{p^\infty}} \rangle [u^{\frac{1}{p^\infty}}] / (\forall k : (u \cdot t^\ell)^{\frac{1}{p^k}} - (f^\sharp)^{\frac{1}{p^k}}). \quad (2)$$

Thus, it is enough to show that the A -algebra

$$C_{\ell,k} := A \langle T^{\frac{1}{p^\infty}} \rangle [u^{\frac{1}{p^k}}] / ((u \cdot t^\ell)^{\frac{1}{p^k}} - (f^\sharp)^{\frac{1}{p^k}})$$

is almost flat over A after reduction modulo t^ϵ for some $\epsilon = \epsilon(\ell, k) > 0$. Choose some $g_k \in A$ such that $g_k^{p^k} = g \pmod{t}$; this is possible as A is perfectoid. Then, for $\epsilon \leq \frac{1}{p^{k+1}}$, we have:

- (1) $(f^\sharp)^{\frac{1}{p^k}} \equiv T^{\frac{1}{p^k}} - g_k \pmod{t^\epsilon}$. Here we use that $\ker(A/t^\delta \xrightarrow{(-)^{p^k}} A/t^\delta) = (t^{\frac{\delta}{p^k}})$ for any $\delta \in \mathbb{N}[\frac{1}{p}] \cap [0, 1]$.
- (2) $t^{\frac{\epsilon}{p^k}} \equiv 0 \pmod{t^\epsilon}$ for all $\ell \in \mathbb{N}$.

Thus, for such an ϵ , we get

$$C_{\ell,k}/t^\epsilon = (A/t^\epsilon)[T^{\frac{1}{p^\infty}}, u^{\frac{1}{p^k}}] / (T^{\frac{1}{p^k}} - g_k).$$

Now the right side above is easily seen to be a free $(A/t^\epsilon)[u^{\frac{1}{p^k}}]$ -module with basis T^i for $0 \leq i < \frac{1}{p^k}$ in $\mathbb{N}[\frac{1}{p}]$. In particular, this is a free A/t^ϵ -module, so we are done. \square

Remark 2.4. Theorem 2.3 is proven in [An2] under a more restrictive setup (but with a stronger conclusion). I am grateful to Scholze for pointing out that the same proof goes through in the above generality.

Remark 2.5. One might worry that the presentations from [Sc1, Lemma 6.4] used above are only valid in the non-derived sense, and thus do not play well with reduction modulo t or t -adic completion. More precisely, one may ask if (2) is also true if one imposes the corresponding relations in the derived sense (i.e., one works with the corresponding Koszul complexes). While answering this question is not necessary for our purposes, the answer is indeed ‘yes’, and we record it here for psychological comfort, especially since such presentations are also important later.

Lemma 2.6. *Let A be an integral perfectoid K° -algebra. Choose $f_1, \dots, f_n, g \in A^\flat$, and set B to be the direct limit of the Koszul complexes $\operatorname{Kos}(A[T_i^{\frac{1}{p^\infty}}]; (g^\sharp \cdot T_i)^{\frac{1}{p^k}} - (f_i^\sharp)^{\frac{1}{p^k}})$. Then the Koszul complex $\operatorname{Kos}(B; t)$ is almost discrete. Thus, the derived t -adic completion of B is almost isomorphic to the perfectoid algebra $A \langle \frac{f_1}{g}, \dots, \frac{f_n}{g} \rangle$.*

⁶The ring A_∞ defined here might not be integral perfectoid, but is almost isomorphic to one (by passing to $(A_\infty)_*$), so we ignore the distinction.

Proof. Note that $A[T_i^{\frac{1}{p^\infty}}]$ has no t -torsion. Thus, the complex $\text{Kos}(B; t)$ is identified with

$$M := \text{colim}_m \left(\text{Kos}(A/t[T_i^{\frac{1}{p^\infty}}]; (g^\# \cdot T_i)^{\frac{1}{p^m}} - (f_i^\#)^{\frac{1}{p^m}} \right)$$

since, at level m , freely imposing the relations $t = 0$ and $(g^\# \cdot T_i)^{\frac{1}{p^m}} - (f_i^\#)^{\frac{1}{p^m}} = 0$ in the derived sense on the ring $A[T_i^{\frac{1}{p^\infty}}]$ can be done in any order. But now M looks the same for both A and A^b , so we may assume that A has characteristic p (and so $f_i = f_i^\#, g = g^\#$). In this case, M identifies with $\text{Kos}(R; t)$, where

$$R := \text{colim}_m \left(\text{Kos}(A[T_i^{\frac{1}{p^\infty}}]; (g \cdot T_i)^{\frac{1}{p^m}} - (f_i)^{\frac{1}{p^m}} \right).$$

But R is discrete: it is the perfection of the derived ring $\text{Kos}(A[T_i^{\frac{1}{p^\infty}}]; g \cdot T_i - f)$, which is always discrete by [BS, Lemma 3.16 or Proposition 5.6]. As $M \simeq \text{Kos}(R; t)$, we are reduced to showing that the t -torsion of R is almost zero. But this follows from perfectness: if $\alpha \in R$ and $t \cdot \alpha = 0$, then $t \cdot \alpha^{p^n} = 0$ for all $n \geq 0$, which, by perfectness, gives $t^{\frac{1}{p^n}} \cdot \alpha = 0$ for all $n \geq 0$, so α is almost zero. \square

In particular, all operations in the proof of Theorem 2.3 can be interpreted in the derived sense.

3. ALMOST-PRO-ZERO MODULES

We introduce the relevant notion of almost mathematics in the pro-category necessary for Theorem 1.5.

Notation 3.1. Let A be a ring equipped with a nonzerodivisor t together with a specified collection $\{t^{\frac{1}{p^k}}\}$ of compatible p -power roots. All occurrences of almost mathematics in this section are with respect to $t^{\frac{1}{p^\infty}}$.

There is an intrinsic notion of almost mathematics of pro- A -modules: one might simply work with pro-objects in the almost category. For example, a projective system $\{M_n\}_{n \geq 1}$ of A -modules is ‘almost-zero’ as a pro-object if for any $n \geq 1$, there exists some $m = m(n) \geq n$ such that the map $M_m \rightarrow M_n$ has image annihilated by $t^{\frac{1}{p^k}}$ for all k . This intrinsic notion is too strong for our purposes, and we use the following weakening, where m depends on k :

Definition 3.2. A pro- A -module $\{M_n\}_{n \geq 1}$ is said to be *almost-pro-zero* if for any $k \geq 0$ and any $n \geq 1$, there exists some $m = m(n, k) \geq n$ such that $\text{im}(M_m \rightarrow M_n)$ is killed by $t^{\frac{1}{p^k}}$; equivalently, for each $k \geq 0$, the map $\{M_n[t^{\frac{1}{p^k}}]\}_{n \geq 1} \rightarrow \{M_n\}_{n \geq 1}$ is a pro-isomorphism in the usual sense. A map of pro-objects in $D^b(A)$ is said to be an *almost-pro-isomorphism* if the cohomology groups of cones form an almost-pro-zero system.

The next few lemmas record the stability properties of this notion:

Lemma 3.3. *If $\{M_n\}_{n \geq 1}$ is an almost-pro-zero pro- A -module, then the complex $R \lim(\{M_n\}_{n \geq 1})$ is almost zero, i.e., it has almost zero cohomology groups.*

Proof. Fix $k \geq 0$. Then the inclusion $\{M_n[t^{\frac{1}{p^k}}]\}_{n \geq 1} \rightarrow \{M_n\}_{n \geq 1}$ is a pro-isomorphism, so both sides have the same $R \lim$. In particular, the cohomology groups of $R \lim(\{M_n\}_{n \geq 1})$ are killed by $t^{\frac{1}{p^k}}$. \square

Lemma 3.4. *If $\{N_n\}_{n \geq 1} \rightarrow \{M_n\}_{n \geq 1}$ is an almost-pro-isomorphism in $D^b(A)$, then $R \lim(\{N_n\}_{n \geq 1}) \rightarrow R \lim(\{M_n\}_{n \geq 1})$ is an almost isomorphism.*

Proof. This follows by applying Lemma 3.3 to the cone. \square

Lemma 3.5. *If $\{M_n\}_{n \geq 1}$ is an almost-pro-zero A -module, and $F : \text{Mod}_A \rightarrow \text{Mod}_A$ is an A -linear functor, then $\{F(M_n)\}_{n \geq 1}$ is also almost-pro-zero.*

Proof. Fix $k \geq 0, n \geq 1$. Then $M_m \rightarrow M_n$ factors over $M_n[t^{\frac{1}{p^k}}] \subset M_n$ for some $m \geq n$. But then $F(M_m) \rightarrow F(M_n)$ factors over $F(M_n[t^{\frac{1}{p^k}}]) \rightarrow F(M_n)$, and hence over $F(M_n)[t^{\frac{1}{p^k}}] \hookrightarrow F(M_n)$, by the A -linearity of F . \square

4. A QUANTITATIVE FORM OF THE RIEMANN EXTENSION THEOREM

Notation 4.1. Let A be an integral perfectoid K° -algebra with associated perfectoid space $X := \mathrm{Spa}(A[\frac{1}{t}])$. Fix an element $g \in A$ that admits a compatible system of p -power roots $g^{\frac{1}{p^k}}$. Assume⁷ that g is a nonzerodivisor modulo t^m in the almost sense (with respect to $t^{\frac{1}{p^\infty}}$).

In this section, we prove Theorem 1.5. Thus, we study the rings $A\langle \frac{t^n}{g} \rangle := \mathcal{O}_X^+(X\langle \frac{t^n}{g} \rangle)$ and their variation with n . More precisely, we show the following quantitative form of Scholze's Hebbbarkeitssatz [Sc2, Proposition II.3.2]:

Theorem 4.2. *For each $m \geq 0$, the natural map*

$$\{A/t^m\}_{n \geq 1} \rightarrow \{A\langle \frac{t^n}{g} \rangle / t^m\}_{n \geq 1}$$

of pro-systems has an almost zero kernel (with respect to $t^{\frac{1}{p^\infty}}$), and an almost-pro-zero cokernel (with respect to $(tg)^{\frac{1}{p^\infty}}$). In particular, it is an almost-pro-isomorphism with respect to $(tg)^{\frac{1}{p^\infty}}$.

To prove this theorem, it is convenient to work with the explicit presentations for $A\langle \frac{t^n}{g} \rangle$ given by Scholze's theory. Thus, for each pair of integers integer $n, k \geq 1$, define the algebras

$$A_{k,n} := A[u_n^{\frac{1}{p^k}}] / ((u_n \cdot g)^{\frac{1}{p^k}} - t^{\frac{n}{p^k}}).$$

Viewing $u_n^{\frac{1}{p^k}} = (\frac{t^n}{g})^{\frac{1}{p^k}}$ defines transition maps $A_{k,n} \rightarrow A_{k+1,n}$ and $A_{k,n+1} \rightarrow A_{k,n}$. Set

$$A_n := \mathrm{colim}_k A_{k,n} = A[u_n^{\frac{1}{p^\infty}}] / (\forall k : (u_n \cdot g)^{\frac{1}{p^k}} - t^{\frac{n}{p^k}}).$$

By [Sc1, Lemma 6.4], the obvious map gives an almost isomorphism (with respect to $t^{\frac{1}{p^\infty}}$)

$$A_n / t^m \simeq A\langle \frac{t^n}{g} \rangle / t^m.$$

As k and n vary, the rings introduced above fit into the following commutative diagram:

$$\begin{array}{ccccccc} A[u_1]/(u_1 \cdot g - t) & \longrightarrow & A[u_1^{\frac{1}{p}}]/((u_1 \cdot g)^{\frac{1}{p}} - t^{\frac{1}{p}}) & \longrightarrow & \dots & \longrightarrow & A[u_1^{\frac{1}{p^k}}]/((u_1 \cdot g)^{\frac{1}{p^k}} - t^{\frac{1}{p^k}}) \longrightarrow \dots \\ \uparrow & & \uparrow & & & & \uparrow \\ A[u_2]/(u_2 \cdot g - t^2) & \longrightarrow & A[u_2^{\frac{1}{p}}]/((u_2 \cdot g)^{\frac{1}{p}} - t^{\frac{2}{p}}) & \longrightarrow & \dots & \longrightarrow & A[u_2^{\frac{1}{p^k}}]/((u_2 \cdot g)^{\frac{1}{p^k}} - t^{\frac{2}{p^k}}) \longrightarrow \dots \\ \uparrow & & \uparrow & & & & \uparrow \\ \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots \\ \uparrow & & \uparrow & & & & \uparrow \\ A[u_n]/(u_n \cdot g - t^n) & \longrightarrow & A[u_n^{\frac{1}{p}}]/((u_n \cdot g)^{\frac{1}{p}} - t^{\frac{n}{p}}) & \longrightarrow & \dots & \longrightarrow & A[u_n^{\frac{1}{p^k}}]/((u_n \cdot g)^{\frac{1}{p^k}} - t^{\frac{n}{p^k}}) \longrightarrow \dots \\ \uparrow & & \uparrow & & & & \uparrow \\ \dots & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & \dots \end{array}$$

Our interest is in the pro-object formed by taking colimits in the horizontal direction. To access this, we begin by observing that multiplication by $g^{\frac{1}{p^k}}$ nullifies the difference between the k -th column above and the ones to its right:

Lemma 4.3. *Fix an integer $k \geq 0$. Then, for any $\ell \geq k$ and any $n \geq 1$, the canonical map $A_{k,n} \rightarrow A_{\ell,n}$ has cokernel killed by $g^{\frac{1}{p^k}}$. In particular, the map $A_{k,n} \rightarrow A_n$ also has cokernel killed by $g^{\frac{1}{p^k}}$.*

⁷This assumption is not actually necessary, and can be dropped *a posteriori*; see Remark 4.7.

Proof. It suffices to show that $g^{\frac{1}{p^k}} \cdot u_n^{\frac{i}{p^\ell}} \in \text{im}(A_{k,n} \rightarrow A_{\ell,n})$ for any $i \in \mathbb{N}$. After renumbering, we may assume $k = 0$, so we must check that $g \cdot u_n^{\frac{i}{p^\ell}} \in \text{im}(A_{0,n} \rightarrow A_{\ell,n})$. Write $i = r + p^\ell q$ with $r, q \geq 0$ and $0 \leq r < p^\ell$. Then

$$g \cdot u_n^{\frac{i}{p^\ell}} = g \cdot u_n^{\frac{r}{p^\ell}} \cdot u_n^q = g^{1-\frac{r}{p^\ell}} \cdot (g \cdot u_n)^{\frac{r}{p^\ell}} \cdot u_n^q = g^{1-\frac{r}{p^\ell}} \cdot t^{\frac{rn}{p^\ell}} \cdot u_n^q \in A_{\ell,n},$$

which is obviously in $\text{im}(A_{0,n} \rightarrow A_{\ell,n})$. \square

Next, we observe that each column is essentially constant:

Lemma 4.4. *Fix integers k and m , both ≥ 0 . Then, if $\ell \geq p^k \cdot m$, then, for each n , the transition map*

$$A_{k,n+\ell}/t^m \rightarrow A_{k,n}/t^m$$

has image contained in the image of the obvious map $A/t^m \rightarrow A_{k,n}/t^m$.

Proof. This A -algebra map sends the generator $u_{n+\ell}^{\frac{1}{p^k}}$ to $u_n^{\frac{1}{p^k}} \cdot t^{\frac{\ell}{p^k}}$, which vanishes modulo t^m if $\ell \geq p^k \cdot m$. \square

The previous two observations combine to prove most of the theorem:

Lemma 4.5. *Fix an integer $m \geq 0$. Consider the canonical map*

$$\{A/t^m\}_{n \geq 1} \rightarrow \{A_n/t^m\}_{n \geq 1}$$

of pro-systems. The cokernel $\{Q_n\}_{n \geq 1}$ of this map is almost-pro-zero with respect to $g^{\frac{1}{p^\infty}}$.

Proof. Fix integers $k \geq 0$ and $n \geq 1$. We must show that there exists some $\ell = \ell(n, k)$ such that the composite map

$$A_{n+\ell}/(t^m, A) \rightarrow A_n/(t^m, A)$$

has image annihilated by $g^{\frac{1}{p^k}}$. Using the factorisation $A \rightarrow A_{k,n} \rightarrow A_n$, the map above fits into the following diagram

$$\begin{array}{ccccccc} A_{k,n}/(t^m, A) & \longrightarrow & A_n/(t^m, A) & \longrightarrow & A_n/(t^m, A_{k,n}) & \longrightarrow & 0 \\ \alpha \uparrow & & \beta \uparrow & & \gamma \uparrow & & \\ A_{k,n+\ell}/(t^m, A) & \longrightarrow & A_{n+\ell}/(t^m, A) & \longrightarrow & A_{n+\ell}/(t^m, A_{k,n+\ell}) & \longrightarrow & 0 \end{array}$$

with exact rows. For $\ell \geq p^k \cdot m$, we have $\alpha = 0$ by Lemma 4.4. Moreover, both the source and target of γ are killed by $g^{\frac{1}{p^k}}$ by Lemma 4.3. A diagram chase shows that β has image annihilated by $g^{\frac{1}{p^k}}$, as wanted. \square

We also need to identify the elements in A that become trivial in the tower $\{A_n\}_{n \geq 1}$:

Lemma 4.6. *Fix an integer n . The kernel of the canonical map $A/t^m \rightarrow A_n/t^m$ is almost zero with respect to $t^{\frac{1}{p^\infty}}$.*

Proof. It is enough to show that $A/t^m \rightarrow A_{k,n}/t^m := (A/t^m)[u_n^{\frac{1}{p^k}}]/((u_n \cdot g)^{\frac{1}{p^k}} - t^{\frac{n}{p^k}})$ is almost injective for each $k \geq 0$. But $t^{\frac{n}{p^k}}$ is nilpotent, while $(u_n \cdot g)^{\frac{1}{p^k}}$ is almost a nonzerodivisor since g is so; the claim follows. \square

Proof of Theorem 4.2. For each $m \geq 0$, the natural map

$$\{A/t^m\}_{n \geq 1} \rightarrow \{A_n/t^m\}_{n \geq 1}$$

of pro-systems has an almost zero kernel (with respect to $t^{\frac{1}{p^\infty}}$) by Lemma 4.6, and an almost-pro-zero cokernel (with respect to $g^{\frac{1}{p^\infty}}$) by Lemma 4.5. Scholze's almost isomorphism $A_n/t^m \simeq A\langle \frac{t^n}{g} \rangle/t^m$ then finishes the proof. \square

Remark 4.7. The assumption that g is a nonzerodivisor modulo t^m in Notation 4.1 can be dropped without affecting the conclusion of Theorem 4.2. Indeed, consider first the universal case $R := K^\circ \langle T^{\frac{1}{p^\infty}} \rangle$ with $g = T$. This falls under the case that is already treated, so we have an almost-pro-isomorphism

$$\{R/t^m\}_{n \geq 1} \rightarrow \{R\langle \frac{t^n}{T} \rangle/t^m\}_{n \geq 1}$$

with respect to $(tT)^{\frac{1}{p^\infty}}$. For general A and g , there is a unique map $R \rightarrow A$ carrying $T^{\frac{1}{p^k}}$ to $g^{\frac{1}{p^k}}$ for all k . By base change, we have an almost-pro-isomorphism

$$\{A/t^m\}_{n \geq 1} \rightarrow \{R\langle \frac{t^n}{T} \rangle \otimes_R^L A/t^m\}_{n \geq 1}$$

with respect to $(tg)^{\frac{1}{p^\infty}}$. The explicit description of [Sc1, Lemma 6.4] shows that

$$R\langle \frac{t^n}{T} \rangle \otimes_R A/t^m \xrightarrow{\sim} A\langle \frac{t^n}{g} \rangle / t^m.$$

In particular, applying H^0 to the almost-pro-isomorphism above gives the desired statement.

5. THE DIRECT SUMMAND CONJECTURE

In this section, we prove Theorem 1.1. We first sketch a standard reduction. Fix a perfect field k of characteristic p , and let $W = W(k)$. We say that a W -algebra R is *p -adically formally smooth* if R is p -adically complete, flat, and R/p is smooth over $k = W/p$; equivalently, R is the p -adic completion of a smooth W -algebra [EI].

Lemma 5.1. *If the conclusion of Theorem 1.1 holds for all p -adically formally smooth W -algebras A_0 and for all possible choices of W , then it holds in general.*

Proof sketch. We may assume $A_0 = W[[x_1, \dots, x_d]]$ by [Ho2, Theorem 6.1]. Popescu's approximation theorem [SP, Tag 07GC] then allows us to assume A_0 is smooth over W . As there is no obstruction to splitting in characteristic 0, the relevant Ext^1 -class is p -torsion, so we may pass to completion to assume that A_0 is p -adically formally smooth. \square

Henceforth, we work with p -adically formally smooth A_0 . Fix the following notation for the rest of this section.

Notation 5.2. Let A_0 be a p -adically formally smooth W -algebra. Fix a finite extension $A_0 \rightarrow B_0$. Choose $g \in A_0$ such that $g \in A_0/p$ is a nonzerodivisor, and that $A_0[\frac{1}{pg}] \rightarrow B_0[\frac{1}{pg}]$ is étale. By [Ke], there is a finite étale map $W[\widehat{x_1, \dots, x_d}] \rightarrow A_0$, where the completion on the left side is p -adic. Set $A_{\infty,0}$ to be the p -adic completion of $A[p^{\frac{1}{p^\infty}}, x_i^{\frac{1}{p^\infty}}]$, so $A_{\infty,0}$ is an integral perfectoid K° -algebra, where K is the perfectoid field $W[p^{\frac{1}{p^\infty}}][\frac{1}{p}]$. Finally, let $A_{\infty,0} \rightarrow A_\infty$ be the extension provided by Theorem 2.3 applied to $A_{\infty,0}$ and the element g ; thus, A_∞ is integral perfectoid, and $A_{\infty,0} \rightarrow A_\infty$ is almost faithfully flat modulo p with respect to $p^{\frac{1}{p^\infty}}$.

With the above notation, our goal is to prove:

Theorem 5.3. *The map $A_0 \rightarrow B_0$ of A_0 -modules is split.*

Proof. Consider the canonical exact triangle

$$A_0 \rightarrow B_0 \rightarrow Q_0$$

of A_0 -modules. The boundary map $\alpha_0 \in \text{Hom}_{A_0}(Q_0, A_0[1])$ is the obstruction to this sequence being split. We would like this show this obstruction vanishes. We change subscript to denote derived base change to either $A_{\infty,0}$ or A_∞ ; for example, $Q_{\infty,0} := Q_0 \otimes_{A_0}^L A_{\infty,0}$, $\alpha_\infty := \alpha_0 \otimes_{A_0}^L A_\infty$, etc.

First, it suffices to show that $\alpha_0/p^m \in \text{Hom}_{A_0}(Q_0, A_0/p^m[1])$ vanishes for all $m \gg 0$. Indeed, we have

$$\text{Hom}_{A_0}(Q_0, A_0[1]) \simeq \lim_m \text{Hom}_{A_0}(Q_0, A_0/p^m[1]),$$

as A_0 is p -adically complete and $\{\text{Hom}_{A_0}(Q_0, A_0/p^m)\}_{m \geq 1}$ has vanishing \lim^1 by noetherianness.

Choose $m \geq 3$ such that $\alpha_0/p^m \neq 0$, so $\text{Ann}_{A_0/p^m}(\alpha_0/p^m) \neq A_0/p^m$; if no such m exists, then $\alpha_0/p^m = 0$ for all m , so we are done. Otherwise, by Krull's theorem, there exists $k \geq 0$ such that $p^2g \notin \text{Ann}_{A_0/p^m}(\alpha_0/p^m)^{p^k}$; here we use that $m \geq 3$ and that $0 \neq g \in A_0/p$. By faithful flatness of $A_0/p^m \rightarrow A_{\infty,0}/p^m$ and almost faithful flatness of $A_{\infty,0}/p^m \rightarrow A_\infty/p^m$ with respect to $p^{\frac{1}{p^\infty}}$, we get $pg \notin \text{Ann}_{A_\infty/p^m}(\alpha_\infty/p^m)^{p^k}$, so $(pg)^{\frac{1}{p^k}} \notin \text{Ann}_{A_\infty/p^m}(\alpha_\infty/p^m)$; here we lose a power of p in passing to almost mathematics. It is thus enough (via contradiction) to show that $\alpha_\infty/p^m \in \text{Hom}_{A_\infty}(Q_\infty, A_\infty/p^m[1])$ is almost zero with respect to $(pg)^{\frac{1}{p^\infty}}$.

Consider the tower $\{A_\infty \langle \frac{p^n}{g} \rangle\}$ from §4. As g divides p^n in $A_\infty \langle \frac{p^n}{g} \rangle$, the base change $A_\infty \langle \frac{p^n}{g} \rangle \rightarrow B_0 \otimes_{A_0}^L A_\infty \langle \frac{p^n}{g} \rangle$ of $A_0 \rightarrow B_0$ is finite étale after inverting p . Almost purity [Sc1, Theorem 7.9 (iii)] then implies that this base change can be dominated by an almost finite étale cover of $A_\infty \langle \frac{p^n}{g} \rangle$, and is thus almost split with respect to $p^{\frac{1}{p^\infty}}$ (see [Bh2, Lemma 2.7]). The same then holds modulo p^m , so the image of α_∞/p^m under

$$\text{can} : \text{Hom}_{A_\infty}(Q_\infty, A_\infty/p^m[1]) \rightarrow \lim_n \text{Hom}_{A_\infty}(Q_\infty, A_\infty \langle \frac{p^n}{g} \rangle / p^m[1])$$

is almost zero with respect to $p^{\frac{1}{p^\infty}}$. It is now enough to show that the above map is an almost isomorphism with respect to $(pg)^{\frac{1}{p^\infty}}$. By Theorem 4.2, the only obstruction is \lim^1 of $\{\text{Hom}_{A_\infty}(Q_\infty, A_\infty \langle \frac{p^n}{g} \rangle / p^m)\}_{n \geq 1}$. This pro-system is almost-pro-isomorphic to a constant pro-system by Lemma 3.5 and Theorem 4.2, so we are done by Lemma 3.4. \square

Remark 5.4. The proof given above goes through for any noetherian ring A_0 that admits a faithfully flat extension which is integral perfectoid. Thus, one may ask: does this condition characterize regularity? In other words, is there a p -adic analog of Kunz's theorem characterizing regularity in characteristic p as the flatness of Frobenius?

6. THE DERIVED DIRECT SUMMAND CONJECTURE

The goal of this section is to prove Theorem 1.2

Theorem 6.1. *Let A_0 be a regular noetherian ring, and let $f_0 : X_0 \rightarrow \text{Spec}(A_0)$ be a proper surjective map. Then the map $A_0 \rightarrow R\Gamma(X_0, \mathcal{O}_{X_0})$ splits in $D(A_0)$.*

Proof. By [Bh1], we may assume that A_0 does not contain a field. Assume first that A_0 is p -adically formally smooth. By taking the closure of a suitable generically defined multisection, we may assume that X_0 is integral and f_0 is generically finite. Then we can choose $g \in A_0$ such that f_0 is finite étale after inverting pg . Construct $A_{\infty,0}$ and A_∞ as Notation 5.2. Repeating the argument in the proof of Theorem 5.3, we must show that for fixed $m, n \geq 1$, the map

$$A_\infty \langle \frac{p^n}{g} \rangle / p^m \rightarrow R\Gamma(X_0, \mathcal{O}_{X_0}) \otimes_{A_0}^L A_\infty \langle \frac{p^n}{g} \rangle / p^m$$

is almost split with respect to $p^{\frac{1}{p^\infty}}$. As the base change $X_0 \times_{\text{Spec}(A_0)} \text{Spec}(A_\infty \langle \frac{p^n}{g} \rangle) \rightarrow \text{Spec}(A_\infty \langle \frac{p^n}{g} \rangle)$ is proper and finite étale after inverting p (as g divides p^n on the base), Proposition 6.2 and a diagram chase finish the proof.

For general regular local rings, one first reduces to the case where A_0 is a complete noetherian regular local ring with perfect residue field. By [Sh, Proposition 4.9] or [An1, Example 3.4.6 (3)], there exists a faithfully flat extension $A_0 \rightarrow A_{\infty,0}$ with $A_{\infty,0}$ perfectoid in a generalized sense. Theorem 2.3 continues to hold for such $A_{\infty,0}$, so the rest of the argument goes through; we omit the details. \square

The following special case of Theorem 6.1 is the crucial one:

Proposition 6.2. *Let A be an integral perfectoid K° -algebra, and set $S = \text{Spec}(A)$. Let $f : Y \rightarrow S$ be a proper morphism such that $f[\frac{1}{p}]$ is finite étale. Then $A \rightarrow R\Gamma(Y, \mathcal{O}_Y)$ is almost split.*

Proof. Let $B = H^0(Y, \mathcal{O}_Y)$, so B is an integral extension of A which is finite étale after inverting p , and Y is naturally a B -scheme. Almost purity [Sc1, Theorem 7.9 (iii)] gives a map $B \rightarrow C$ which is an isomorphism after inverting p such that the induced map $A \rightarrow C$ is an almost finite étale cover. In particular, C is integral perfectoid, and $A \rightarrow C$ is almost split. Thus, on replacing A with C and Y with $Y \otimes_B C$, we may assume that $f[\frac{1}{p}]$ is an isomorphism. But then the p -adic completion \widehat{f} of f can be dominated by an admissible blowup of S . Set $S_\eta = \text{Spa}(A[\frac{1}{p}])$ to be the associated affinoid perfectoid space, so the natural map $(S_\eta, \mathcal{O}_{S_\eta}^+) \rightarrow S$ factors through every admissible blowup of \widehat{S} . In particular, it factors as

$$(S_\eta, \mathcal{O}_{S_\eta}^+) \rightarrow Y \rightarrow S.$$

Taking cohomology of the structure sheaf gives

$$A \xrightarrow{b} R\Gamma(Y, \mathcal{O}_Y) \xrightarrow{a} R\Gamma(S_\eta, \mathcal{O}_{S_\eta}^+).$$

Now $a \circ b$ is an almost isomorphism by Scholze's vanishing theorem [Sc1, Proposition 6.14], so b is almost split. \square

REFERENCES

- [An1] Y. André, *La lemme d'Abhyankar perfectoid*, preprint, 55 pp.
- [An2] Y. André, *La conjecture du facteur direct*, preprint, 15 pp.
- [Ba] W. Bartenwerfer, *Der erste Riemannsche Hebbarkeitssatz im nichtarchimedischen Fall*, J. reine und angew. Math. 286-287 (1976), 144-163.
- [Bh1] B. Bhatt, *Derived splinters in positive characteristic*, Compos. Math. 148 (2012), no. 6, 1757-1786.
- [Bh2] B. Bhatt, *Almost direct summands*, Nagoya math. J. 214 (2014), 195-204.
- [Bh3] B. Bhatt, *p -divisibility for coherent cohomology*, Forum Math. Sigma 3 (2015), e15, 27 pp.
- [BS] B. Bhatt and P. Scholze, *Projectivity of the Witt vector affine Grassmannian*, arXiv eprint:1507.06490, 56 pp.
- [El] R. Elkik, *Solutions d'équations à coefficients dans un anneau hensélien*, Ann. Sci. Ecole Norm. Sup. (4) 6 (1973), 553-603 (1974).
- [Fa1] G. Faltings, *p -adic Hodge theory*, J. Amer. Math. Soc. 1, 1 (1988), 255-299.
- [Fa2] G. Faltings, *Crystalline cohomology and p -adic Galois representations*, in Algebraic analysis, geometry, and number theory, Baltimore, MD: Johns Hopkins University Press, pp. 25-80.
- [Fa3] G. Faltings, *Integral crystalline cohomology over very ramified valuation rings*, J. Amer. Math. Soc. 12, 1 (1999), 117-144.
- [Fa4] G. Faltings, *Almost étale extensions*, Astérisque 279 (2002), 185-270.
- [GR] O. Gabber and L. Ramero, *Almost ring theory*, Lecture Notes in Math. 1800, Springer (2003).
- [He] R. Heitmann, *The direct summand conjecture in dimension 3*, Ann. of Math., 156 (2002), 695-712.
- [Ho1] M. Hochster, *Contracted ideals from integral extensions of regular rings*, Nagoya Math. J., 51 (1973), 25-43.
- [Ho2] M. Hochster, *Canonical elements in local cohomology modules and the direct summand conjecture*, J. of Algebra 84 (1983), 503-553.
- [Ho3] M. Hochster, *Homological conjectures, old and new*, Illinois J. Math. 51 (2007), 151-169.
- [Ke] K. Kedlaya, *More étale covers of affine spaces in positive characteristic*, J. Algebraic Geom., 14(1):187-192, 2005.
- [KL] K. Kedlaya and R. Liu, *Relative p -adic Hodge theory: Foundations*, Astérisque No. 371 (2015), 239 pp.
- [Ko] S. Kovács, *A characterization of rational singularities* Duke Math. J. 102 (2000).
- [Sc1] P. Scholze, *Perfectoid spaces*, Publ. Math. Inst. Hautes études Sci. 116 (2012), 245-313.
- [Sc2] P. Scholze, *On torsion in the cohomology of locally symmetric spaces*, Ann. of Math. (2) 182 (2015), no. 3, 945-1066.
- [Sh] K. Shimomoto, *An application of the almost purity theorem to the homological conjectures*, J. pure applied algebra 220 (2014).
- [SP] *The Stacks Project*. Available at <http://stacks.math.columbia.edu>.
- [Ta] J. Tate, *p -divisible groups*, in Proc. Conf. Local Fields (Driebergen, 1966), Springer, Berlin, pages 158-183, 1967.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR